Kuwait University — **Dept. of Math. & Comp. Sci.**

Math 101 — First Exam

Date: March 25, 2008 — **Duration: 90 minutes**

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed

Answer the following questions:

1. Evaluate the following limits, if they exist:

(a) $$\lim_{x\to 1} \frac{x^3-1}{x^2-3x+2}.$$ (3 pts.)

(b) $$\lim_{x\to 1} \left[x^5 + (x^2-2x+1)\sin\left(\frac{1}{x-1}\right)\right].$$ (3 pts.)

2. Find the vertical and horizontal asymptotes, if any, for the graph of

$$f(x) = \frac{x^2 + x\sqrt{x^2+1}}{(x+1)^2}.$$ (4 pts.)

3. Let

$$f(x) = \begin{cases} A + \dfrac{3|x-1|}{x^2+x-2} & \text{, if } x<1, x\neq -2, \\ B & \text{, if } x = 1, \\ \sqrt{2x-1} & \text{, if } x>1. \end{cases}$$ (4pts.)

Find the values of A and B, so that f is continuous at $x=1$.

4. (a) State The Intermediate Value Theorem. (1 pt.)

(b) Use The Intermediate Value Theorem to show that there is a real solution of the equation $(x+1)\sqrt{x-1} = 1$. (3 pts.)

5. Let $f(x) = \dfrac{|x|(x-3)}{x^3-9x}$. Classify the types of discontinuity of f as removable, jump, or infinite. (4 pts.)

6. Use the definition of the derivative to find $f'(2)$, where $f(x) = \sqrt{x+2} - 1$. (3 pts.)

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1. (a) $\lim\limits_{x\to 1}\frac{x^3-1}{x^2-3x+2} = \lim\limits_{x\to 1}\frac{(x-1)\left(x^2+x+1\right)}{(x-1)(x-2)} = \lim\limits_{x\to 1}\frac{x^2+x+1}{x-2} = \boxed{-3}$

(b) For $x \neq 1$, $-1 \leq \sin\frac{1}{x-1} \leq 1 \Longrightarrow -(x-1)^2 \leq (x-1)^2 \sin\left(\frac{1}{x-1}\right) \leq (x-1)^2$.
Since, $\lim\limits_{x\to 1}(x-1)^2 = 0 = \lim\limits_{x\to 1} -(x-1)^2$, then from The Squeeze Theorem, $\lim\limits_{x\to 1}(x-1)^2\sin\left(\frac{1}{x-1}\right) = 0$.

$\therefore \lim\limits_{x\to 1}\left[x^5 + (x^2-2x+1)\sin\left(\frac{1}{x-1}\right)\right] = \lim\limits_{x\to 1}\left[x^5 + (x-1)^2\sin\left(\frac{1}{x-1}\right)\right] = \lim\limits_{x\to 1}x^5 + \lim\limits_{x\to 1}(x-1)^2\sin\left(\frac{1}{x-1}\right) = 1+0 = \boxed{1}$

2. $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty}\frac{x^2+x\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{x^2+2x+1} = \lim\limits_{x\to\pm\infty}\frac{x^2+x|x|\sqrt{1+\frac{1}{x^2}}}{x^2\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}$

$\lim\limits_{x\to-\infty}\frac{1-\sqrt{1+\frac{1}{x^2}}}{1+\frac{2}{x}+\frac{1}{x^2}} = \boxed{0}$ & $\lim\limits_{x\to\infty}\frac{1+\sqrt{1+\frac{1}{x^2}}}{1+\frac{2}{x}+\frac{1}{x^2}} = 2 \Rightarrow \boxed{y=0 \text{ \& } y=2}$ are $H.A$ for the graph of f.

$\lim\limits_{x\to -1^{\pm}} f(x) = \boxed{-\infty} \Rightarrow \boxed{x=-1}$ is $V.A$ for the graph of f.

3. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}\left[A + \frac{3|x-1|}{x^2+x-2}\right] = \lim\limits_{x\to 1^-}\left[A + \frac{-3(x-1)}{(x-1)(x+2)}\right] = \boxed{A-1}$,
$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}\sqrt{2x-1} = \sqrt{\lim\limits_{x\to 1^+}(2x-1)} = \boxed{1}$ & $f(1) = \boxed{B}$. Since f is continuous at 1, then $\lim\limits_{x\to 1^-} f(x) = f(1) = \lim\limits_{x\to 1^+} f(x) \Longrightarrow \boxed{A=2,\ B=1}$.

4. (b) Let $f(x) = (x+1)\sqrt{x-1} - 1$. f is *continuous* on $[1,2]$, and $f(1) = -1 < 0$ & $f(2) = 2 > 0$. From The Intermediate Value Theorem, f has at least one $c \in (1,2)$ such that $f(c) = 0$. Thus, c is a solution of the equation.

5. $x^3 - 9x = x(x-3)(x+3)$, *Domain* $f = \mathbb{R} - \{-3, 0, 3\}$.

$\lim\limits_{x\to -3^{\pm}} f(x) = \lim\limits_{x\to -3^{\pm}}\frac{|x|(x-3)}{x(x-3)(x+3)} = \boxed{\mp\infty} \Longrightarrow f$ has an <u>*infinite discontinuity*</u> at $x = -3$.

$\lim\limits_{x\to 0^{\pm}} f(x) = \lim\limits_{x\to 0^{\pm}}\frac{\pm x(x-3)}{x(x-3)(x+3)} = \boxed{\pm\frac{1}{3}} \Longrightarrow f$ has a <u>*jump discontinuity*</u> at $x = 0$.

$\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3}\frac{|x|(x-3)}{x(x-3)(x+3)} = \boxed{\frac{1}{6}} \Longrightarrow f$ has a <u>*removable discontinuity*</u> at $x = 3$.

6. $f'(2) = \lim\limits_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2}\frac{\left(\sqrt{x+2}-1\right)-(1)}{x-2} = \lim\limits_{x\to 2}\frac{\sqrt{x+2}-2}{x-2} \times \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim\limits_{x\to 2}\frac{(x+2)-(2)^2}{(x-2)\left[\sqrt{x+2}+2\right]} = \lim\limits_{x\to 2}\frac{1}{\sqrt{x+2}+2} = \boxed{\frac{1}{4}}$